Definitions and key facts for section 4.4

Fact: The unique representation theorem
Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V there exists a *unique* set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

We call these scalars $c_1, \ldots, c_n$ the **coordinates of x relative to the basis** $\mathcal{B}$ (or more simply, the **$\mathcal{B}$-coordinates of x**).

The vector in $\mathbb{R}^n$ consisting of the $\mathcal{B}$-coordinates of $\mathbf{x}$

$$\left[\mathbf{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x relative to** $\mathcal{B}$ or the **$\mathcal{B}$-coordinate vector of x**.

For a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$, we call the matrix

$$P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$

the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

Note for any vector $\mathbf{x}$ in $\mathbb{R}^n$, we have

$$P_{\mathcal{B}}\left[\mathbf{x}\right]_{\mathcal{B}} = \mathbf{x} \text{ and } \left[\mathbf{x}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}.$$

For a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space V, the **coordinate mapping** $\mathbf{x} \mapsto \left[\mathbf{x}\right]_{\mathcal{B}}$ is a one-to-one and onto linear transformation from V to $\mathbb{R}^n$.

We call such a map an *isomorphism* and say that V and $\mathbb{R}^n$ are *isomorphic* vector spaces.